Cyclic Homology.

Assume there is given a cycle $(\Omega^\bullet, d, \int)$, an algebra A, and a dga map $\varphi : \Omega^\bullet A \to \Omega^\bullet$.

Consider a map

$$A^{\otimes n+1} \longrightarrow \mathbb{C}$$

$$(a_0, a_1, \dots, a_n) \longmapsto \int \varphi(a_0)\, d\varphi(a_1) \dots d\varphi(a_n).$$

For the sake of simplicity of the notation we will suppress φ.

E.g. for $n=2$

$$(a_0, a_1, a_2) \longmapsto \int a_0 \, da_1 \, da_2$$

By the graded trace property of $\int$ we have

$$\int a_0 \, da_1 \, da_2 = -\int da_2 \, a_0 \, da_1 .$$

By the odd derivation property of d the latter can be written as

$$-\int d(a_2 a_0 \, da_1) - a_2 \, da_0 \, da_1 .$$

By closedness of $\int$ it becomes

$$\int a_2 \, da_0 \, da_1$$

which is the image of (a_2, a_0, a_1). This implies that our map kills the difference

$$(a_0, a_1, a_2) - (a_2, a_0, a_1).$$

Using a map

$$t : A^{\oplus 3} \longrightarrow A^{\oplus 3}$$
$$(a_0, a_1, a_2) \longmapsto (a_2, a_0, a_1)$$

our map can be equivalently written as

$$A^{\oplus 3} / \mathrm{im}(1-t) \longrightarrow \mathbb{C}.$$

For arbitrary n we get similarly a map

$$A^{\oplus(n+1)} / \mathrm{im}(1-t) \longrightarrow \mathbb{C}$$

where

$$t(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1}).$$

Moreover,

$$(a_0 a_1, a_2, a_3) - (a_0, a_1 a_2, a_3) + (a_0, a_1, a_2 a_3) - (a_3 a_0, a_1, a_2)$$

is mapped into

$$\int a_0 a_1 \, da_2 \, da_3 - a_0 \, d(a_1 a_2) \, da_3 + a_0 \, da_1 \, d(a_2 a_3) - a_3 a_0 \, da_1 \, da_2$$

which by the odd derivation property of d can be written as

$$\int \cancel{a_0 a_1 \, da_2 \, da_3} - \cancel{a_0 \, da_1 \, a_2 \, da_3} - \cancel{a_0 a_1 \, da_2 \, da_3}$$
$$+ a_0 \, da_1 \, da_2 \, a_3 + \cancel{a_0 \, da_1 \, a_2 \, da_3} - a_3 a_0 \, da_1 \, da_2$$

and hence by the graded trace property of $\int$ vanishes

$$\int a_0 da_1 da_2 a_3 - a_3 a_0 da_1 da_2 = 0.$$

For arbitrary n one defines the operator

$$\begin{aligned} b(a_0, \dots, a_n) = {} & (a_0 a_1, a_2, \dots, a_n) - (a_0, a_1 a_2, a_3, \dots, a_n) \\ & + (a_0, a_1, a_2 a_3, \dots, a_n) - \dots + (-1)^{n-1} (a_0, \dots, a_{n-1} a_n) \\ & + (-1)^n (a_n a_0, a_1, \dots, a_{n-1}) \end{aligned}$$

This satisfies $b^2 = 0$ and $b(t-1) = (t-1) b'$, where

$$\begin{aligned} b'(a_0, \dots, a_n) = {} & (a_0 a_1, a_2, \dots, a_n) - (a_0, a_1 a_2, a_3, \dots, a_n) \\ & + (a_0, a_1, a_2 a_3, \dots, a_n) - \dots + (-1)^{n-1} (a_0, \dots, a_{n-1} a_n) \end{aligned}$$

(without coming back with a_n to the front of the sequence), e.g.

$$b^2(a_0, a_1, a_2) = b\big((a_0a_1, a_2) - (a_0, a_1a_2) + (a_2a_0, a_1)\big)$$

$$= \cancel{a_0a_1a_2} - \cancel{a_2a_0a_1} - \cancel{a_0a_1a_2} + \underline{a_1a_2a_0} + \cancel{a_2a_0a_1} - \underline{a_1a_2a_0}$$

$$= 0,$$

$$b(1-t)\,(a_0,a_1,a_2) = b\big((a_0,a_1,a_2) - (a_2,a_0,a_1)\big)$$

$$= \underline{(a_0a_1,a_2)} - (a_0,a_1a_2) + \cancel{(a_2a_0,a_1)}$$

$$- \cancel{(a_2a_0,a_1)} + \underline{(a_2,a_0a_1)} - (a_1a_2,a_0)$$

$$= \underline{(a_0a_1,a_2)} + \underline{(a_2,a_0a_1)} - (a_0,a_1a_2) - (a_1a_2,a_0)$$

$$= (1-t)\big((a_0a_1,a_2) - (a_0,a_1a_2)\big) = (1-t)\,b'(a_0,a_1,a_2).$$

Therefore, $b(\mathrm{im}(1-t)) \subset \mathrm{im}(1-t)$ and hence

$$\left(C_\bullet(A) = A^{\otimes(\bullet+1)}/\mathrm{im}(1-t),\ b\right)$$ is a complex.

If one denotes the homology of this complex by $HC_\bullet(A)$ one sees that our map induces a map

$$HC_\bullet(A) \longrightarrow \mathbb{C}.$$

Theorem. [Hochschild-Kostant-Rosenberg] Let A be a finitely generated commutative algebra s.t. Ω_A^1 is a finitely generated projective A-module. Then there is an isomorphism of vector spaces

$$HC_n(A) \cong \Omega_A^n / d\Omega_A^{n-1} \oplus H_{dR}^{n-2}(A) \oplus H_{dR}^{n-4}(A) \oplus \cdots .$$

Moreover,

the map
$$A^{\otimes(n+1)} \longrightarrow \Omega^n_A,$$
$$(a_0, \ldots, a_n) \longmapsto \frac{1}{n!} a_0 da_1 \ldots da_n$$
induces a map
$$HC_n(A) \longrightarrow H^n_{dR}(A). \qquad \square$$

Exercise 57. Show that an element

$$\sum_{i_0,\dots,i_{2n}} (p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{2n} i_0}) \in A^{\otimes(2n+1)}$$

for an idempotent $p \in M_k(A)$ defines an element in $HC_{2n}(A)$.

Solution.

$$b \sum_{i_0,\dots,i_{2n}} (p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{2n} i_0}) = \sum_{i_0,\dots,i_{2n}} (p_{i_0 i_1} p_{i_1 i_2}, p_{i_2 i_3}, \dots, p_{i_{2n} i_0})$$

$$- \sum_{i_0,\dots,i_n} (p_{i_0 i_1}, p_{i_1 i_2} p_{i_2 i_3}, \dots, p_{i_{2n} i_0}) + \dots$$

$$\dots + (-1)^{n-1} \sum_{i_0,\dots,i_n} (p_{i_0 i_1}, \dots, p_{i_{2n-1} i_{2n}} p_{i_{2n} i_0})$$

$$+ (-1)^n \sum_{i_0,\dots,i_n} (p_{i_{2n} i_0} p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{2n-1} i_{2n}})$$

$$= \sum_{i_0,\dots,i_m} \left(p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{m-1} i_0}\right) - \sum_{i_0,\dots,i_m} \left(p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{m-1} i_0}\right)$$

$$+ \dots + (-1)^{2n} \sum_{i_0,\dots,i_m} \left(p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{m-1} i_0}\right)$$

$$= \sum_{i_0,\dots,i_m} \left(p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{m-1} i_0}\right)$$

$$= (1-t)\,\frac{1}{2} \sum_{i_0,\dots,i_m} \left(p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{m-1} i_{2n}}\right). \quad \blacksquare$$

Exercise 58. Show that for a commutative algebra A the element

$$\left[\frac{(2n)!}{(2\pi i)^n n!} \sum_{i_0,\dots,i_{2n}} (p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{2n} i_0})\right] \in HC_{2n}(A)$$

for an idempotent $p \in M_r(A)$ maps under the map induced by

$$A^{\otimes(2n+1)} \longrightarrow \Omega^{2n}_A$$

$$(a_0, \dots, a_{2n}) \longmapsto \frac{1}{(2n)!} a_0 \, da_1 \dots da_{2n}$$

into $ch_n(p) \in H^{2n}_{dR}(A)$.

Solution.

$$\left[\frac{(2n)!}{n!}\frac{1}{(2\pi i)^n}\sum_{i_0,\dots,i_{2n}}(p_{i_0 i_1}, p_{i_1 i_2}, \dots, p_{i_{2n} i_0})\right] \longmapsto$$

$$\left[\frac{1}{n!}\frac{1}{(2\pi i)^n}\sum_{i_0,\dots,i_{2n}} p_{i_0 i_1}\, dp_{i_1 i_2} \dots dp_{i_{2n} i_0}\right]$$

$$= \left[\frac{1}{n!}\frac{1}{(2\pi i)^n}\operatorname{tr}(p\, dp \dots dp)\right] = ch_n(p).$$

Corollary. The Chern character in the de Rham cohomology is a canonical image of its noncommutative counterpart in cyclic homology.